

# Well-dominated graphs without cycles of lengths 4 and 5

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## Abstract

Let  $G$  be a graph. A set  $S$  of vertices in  $G$  *dominates* the graph if every vertex of  $G$  is either in  $S$  or a neighbor of a vertex in  $S$ . Finding a minimal cardinality set which dominates the graph is an **NP**-complete problem. The graph  $G$  is *well-dominated* if all its minimal dominating sets are of the same cardinality. The complexity status of recognizing well-dominated graphs is not known. We show that recognizing well-dominated graphs can be done polynomially for graphs without cycles of lengths 4 and 5, by proving that a graph belonging to this family is well-dominated if and only if it is well-covered.

Assume that a weight function  $w$  is defined on the vertices of  $G$ . Then  $G$  is *w-well-dominated* if all its minimal dominating sets are of the same weight. We prove that the set of weight functions  $w$  such that  $G$  is *w-well-dominated* is a vector space, and denote that vector space by  $WWD(G)$ . We prove that  $WWD(G)$  is a subspace of  $WCW(G)$ , the vector space of weight functions  $w$  such that  $G$  is *w-well-covered*. We provide a polynomial characterization of  $WWD(G)$  for the case that  $G$  does not contain cycles of lengths 4, 5, and 6.

**Keywords:** vector space, minimal dominating set, maximal independent set, well-dominated graph, well-covered graph

## 1 Introduction

### 1.1 Definitions and Notations

Throughout this paper  $G$  is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set  $V(G)$  and edge set  $E(G)$ .

Cycles of  $k$  vertices are denoted by  $C_k$ . When we say that  $G$  does not contain  $C_k$  for some  $k \geq 3$ , we mean that  $G$  does not admit subgraphs isomorphic to  $C_k$ . It is important to mention that these subgraphs are not necessarily induced. Let  $\mathcal{G}(\widehat{C_{i_1}}, \dots, \widehat{C_{i_k}})$  be the family of all graphs which do not contain  $C_{i_1}, \dots, C_{i_k}$ .

Let  $u$  and  $v$  be two vertices in  $G$ . The *distance* between  $u$  and  $v$ , denoted  $d(u, v)$ , is the length of a shortest path between  $u$  and  $v$ , where the length of a path is the number of its edges. If  $S$  is a non-empty set of vertices, then the *distance* between  $u$  and  $S$ , denoted  $d(u, S)$ , is defined by:

$$d(u, S) = \min\{d(u, s) : s \in S\}.$$

For every  $i$ , denote

$$N_i(S) = \{x \in V(G) : d(x, S) = i\},$$

and

$$N_i[S] = \{x \in V(G) : d(x, S) \leq i\}.$$

We abbreviate  $N_1(S)$  and  $N_1[S]$  to be  $N(S)$  and  $N[S]$ , respectively. If  $S$  contains a single vertex,  $v$ , then we abbreviate  $N_i(\{v\})$ ,  $N_i[\{v\}]$ ,  $N(\{v\})$ , and  $N[\{v\}]$  to be  $N_i(v)$ ,  $N_i[v]$ ,  $N(v)$ , and  $N[v]$ , respectively.

For every vertex  $v \in V(G)$ , the *degree* of  $v$  is  $d(v) = |N(v)|$ . Let  $L(G)$  be the set of all vertices  $v \in V(G)$  such that either  $d(v) = 1$  or  $v$  is on a triangle and  $d(v) = 2$ .

## 1.2 Well-Covered Graphs

A set of vertices is *independent* if its elements are pairwise nonadjacent. Define  $D(v) = N(v) \setminus N(N_2(v))$ , and let  $M(v)$  be a maximal independent set of  $D(v)$ . An independent set of vertices is *maximal* if it is not a subset of another independent set. An independent set is *maximum* if  $G$  does not admit an independent set with a bigger cardinality. Denote  $i(G)$  the minimal cardinality of a maximal independent set in  $G$ , where  $\alpha(G)$  is the cardinality of a maximum independent set in  $G$ .

The graph  $G$  is *well-covered* if  $i(G) = \alpha(G)$ , i.e. all maximal independent sets are of the same cardinality. The problem of finding a maximum cardinality independent set  $\alpha(G)$  in an input graph is **NP**-complete. However, if the input is restricted to well-covered graphs, then a maximum cardinality independent set can be found polynomially using the *greedy algorithm*.

Let  $w : V(G) \rightarrow \mathbb{R}$  be a weight function defined on the vertices of  $G$ . For every set  $S \subseteq V(G)$ , define  $w(S) = \sum_{s \in S} w(s)$ . The graph  $G$  is *w-well-covered* if all maximal independent sets are of the same weight. The set of weight functions  $w$  for which  $G$  is *w-well-covered* is a *vector space* [2]. That vector space is denoted  $WCW(G)$  [1].

Since recognizing well-covered graphs is **co-NP**-complete [4] [14], finding the vector space  $WCW(G)$  of an input graph  $G$  is **co-NP**-hard. Finding  $WCW(G)$  remains **co-NP**-hard when the input is restricted to graphs with girth at least 6 [12], and bipartite graphs [12]. However, the problem is polynomially solvable for  $K_{1,3}$ -free graphs [11], and for graphs with a bounded maximal degree [12]. For every graph  $G$  without cycles of lengths 4, 5, and 6, the vector space  $WCW(G)$  is characterized as follows.

**Theorem 1** [10] Let  $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5}, \widehat{C_6})$  be a graph, and let  $w : V(G) \rightarrow \mathbb{R}$ . Then  $G$  is  $w$ -well-covered if and only if one of the following holds:

1.  $G$  is isomorphic to either  $C_7$  or  $T_{10}$  (see Figure 1), and there exists a constant  $k \in \mathbb{R}$  such that  $w \equiv k$ .
2. The following conditions hold:
  - $G$  is isomorphic to neither  $C_7$  nor  $T_{10}$ .
  - For every two vertices,  $l_1$  and  $l_2$ , in the same component of  $L(G)$  it holds that  $w(l_1) = w(l_2)$ .
  - For every  $v \in V(G) \setminus L(G)$  it holds that  $w(v) = w(M(v))$  for some maximal independent set  $M(v)$  of  $D(v)$ .

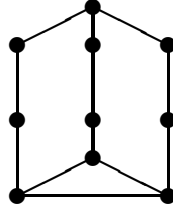


Figure 1: The graph  $T_{10}$ .

Recognizing well-covered graphs is a restricted case of finding  $WCW(G)$ . Therefore, for all families of graphs for which finding  $WCW(G)$  is polynomial solvable, recognizing well-covered graphs is polynomial solvable as well. Recognizing well-covered graphs is **co-NP**-complete for  $K_{1,4}$ -free graphs [3], but it is polynomially solvable for graphs without cycles of lengths 3 and 4 [7], for graphs without cycles of lengths 4 and 5 [8], or for chordal graphs [13].

### 1.3 Well-Dominated Graphs

Let  $S$  and  $T$  be two sets of vertices of the graph  $G$ . Then  $S$  *dominates*  $T$  if  $T \subseteq N[S]$ . The set  $S$  is *dominating* if it dominates all vertices of the graph. A dominating set is *minimal* if it does not contain another dominating set. A dominating set is *minimum* if  $G$  does not admit a dominating set with smaller cardinality. Let  $\gamma(G)$  be the cardinality of a minimum dominating set in  $G$ , and let  $\Gamma(G)$  be the maximal cardinality of a minimal dominating set of  $G$ . If  $\gamma(G) = \Gamma(G)$  then the graph is *well-dominated*, i.e. all minimal dominating sets are of the same cardinality. This concept was introduced in [6], and further studied in [9]. The fact that every maximal independent set is also a minimal dominating set implies that

$$\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$$

for every graph  $G$ . Hence, if  $G$  is not well-covered, then it is not well-dominated.

**Theorem 2** [6] *Every well-dominated graph is well-covered.*

In what follows, our main subject is the interplay between well-covered and well-dominated graphs.

**Problem 3** *WD*

*Input:* A graph  $G$ .

*Question:* Is  $G$  well-dominated?

It is even not known whether the *WD* problem is in **NP** [5]. However, the *WD* problem is polynomial for graphs with girth at least 6 [6], and for bipartite graphs [6]. We prove that a graph without cycles of lengths 4 and 5 is well-dominated if and only if it is well-covered. Consequently, by [8], recognizing well-dominated graphs without cycles of lengths 4 and 5 is a polynomial task.

Let  $w : V(G) \rightarrow \mathbb{R}$  be a weight function defined on the vertices of  $G$ . Then  $G$  is *w-well-dominated* if all minimal dominating sets are of the same weight. Let  $WWD(G)$  denote the set of weight functions  $w$  such that  $G$  is *w-well-dominated*. It turns out that for every graph  $G$ ,  $WWD(G)$  is a vector space. Moreover, if  $G$  is *w-well-dominated* then  $G$  is *w-well-covered*, i.e.,  $WWD(G) \subseteq WCW(G)$ .

**Problem 4** *WWD*

*Input:* A graph  $G$ .

*Output:* The vector space of weight functions  $w$  such that  $G$  is *w-well-dominated*.

Finally, we supply a polynomial characterization of the *WWD* problem, when its input is restricted to  $\mathcal{G}(\widehat{C_4}, \widehat{C_5}, \widehat{C_6})$ .

## 2 Well-Dominated Graphs Without $C_4$ and $C_5$

A vertex  $v$  is *simplicial* if  $N[v]$  is a clique. In [8] the family  $F$  of graphs is defined as follows. A graph  $G$  is in the family  $F$  if there exists  $\{x_1, \dots, x_k\} \subseteq V(G)$  such that  $x_i$  is simplicial for each  $1 \leq i \leq k$ , and  $\{N[x_i] : 1 \leq i \leq k\}$  is a partition of  $V(G)$ . Well-covered graphs without cycles of lengths 4 and 5 are characterized as follows.

**Theorem 5** [8] *Let  $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5})$  be a connected graph. Then  $G$  is well-covered if and only if one of the following holds:*

1.  $G$  is isomorphic to either  $C_7$  or  $T_{10}$ .
2.  $G$  is a member of the family  $F$ .

Actually, under the restriction  $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5})$ , the families of well-covered and well-dominated graphs coincide.

**Theorem 6** *Let  $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5})$  be a connected graph. Then  $G$  is well-dominated if and only if it is well-covered.*

**Proof.** By Theorem 2, if  $G$  is not well-covered then it is not well-dominated.

Assume  $G$  is well-covered, and it should be proved that  $G$  is well-dominated. One can verify that  $\gamma(C_7) = \Gamma(C_7) = 3$ , and  $\gamma(T_{10}) = \Gamma(T_{10}) = 4$ . Therefore,  $C_7$  and  $T_{10}$  are well-dominated.

By Theorem 5, it remains to prove that if  $G$  is a member of  $F$  then it is well-dominated. There exists  $\{x_1, \dots, x_k\} \subseteq V(G)$  such that  $x_i$  is simplicial for each  $1 \leq i \leq k$ , and  $\{N[x_i] : 1 \leq i \leq k\}$  is a partition of  $V(G)$ . Define  $V_i = N[x_i]$  for each  $1 \leq i \leq k$ . Let  $S$  be a minimal dominating set of  $G$ . It is enough to prove that  $|S| = k$ . The fact that  $S$  dominates  $x_i$  implies that  $S \cap V_i \neq \emptyset$ . Assume on the contrary that there exists  $1 \leq i \leq k$  such that  $|V_i \cap S| \geq 2$ . Let  $S' \subset S$  such that  $|S' \cap V_i| = 1$  for each  $1 \leq i \leq k$ . Clearly,  $S'$  dominates the whole graph, which is a contradiction. Therefore,  $|S| = k$ , and  $G$  is well-dominated. ■

If  $G \notin \mathcal{G}(\widehat{C_4})$ , then Theorem 6 does not hold. Let  $n \geq 3$ . Obviously,  $K_{n,n} \in \mathcal{G}(\widehat{C_5})$ , and the cardinality of every maximal independent set of  $K_{n,n}$  is  $n$ . Therefore,  $K_{n,n}$  is well-covered. However, there exists a minimal dominating set of cardinality 2. Therefore,  $K_{n,n}$  is not well-dominated.

If  $G \notin \mathcal{G}(\widehat{C_5})$ , then Theorem 6 does not hold. Let  $G$  be comprised of three disjoint 5-cycles,  $(x_1, \dots, x_5), (y_1, \dots, y_5), (z_1, \dots, z_5)$ , and a triangle  $(x_1, y_1, z_1)$ . Clearly,  $G \in \mathcal{G}(\widehat{C_4})$ , and every maximal independent set contains 2 vertices from each 5-cycle. Hence, the cardinality of every maximal independent set is 6, and  $G$  is well-covered. However,  $G$  is not well-dominated because it contains a minimal dominating set of cardinality 7:  $\{x_1, x_2, x_5, y_3, y_4, z_3, z_4\}$ .

### 3 Weighted Well-Dominated Graphs

**Theorem 7** *Let  $G$  be a graph. Then the set of weight functions  $w : V(G) \rightarrow \mathbb{R}$  such that  $G$  is  $w$ -well-dominated is a vector space.*

**Proof.** Obviously, if  $w_0 \equiv 0$  then  $G$  is  $w_0$ -well-dominated.

Let  $w_1, w_2 : V(G) \rightarrow \mathbb{R}$ , and assume that  $G$  is  $w_1$ -well-dominated and  $w_2$ -well-dominated. Then there exist two constants,  $t_1$  and  $t_2$ , such that  $w_1(S) = t_1$  and  $w_2(S) = t_2$  for every minimal dominating set  $S$  of  $G$ . Let  $\lambda \in \mathbb{R}$ , and let  $w : V(G) \rightarrow \mathbb{R}$  be defined by  $w(v) = w_1(v) + \lambda w_2(v)$  for every  $v \in V(G)$ . Then for every minimal dominating set  $S$  it holds that

$$w(S) = \sum_{s \in S} w(s) = \sum_{s \in S} (w_1(s) + \lambda w_2(s)) = \sum_{s \in S} w_1(s) + \lambda \sum_{s \in S} w_2(s) = t_1 + \lambda t_2,$$

and  $G$  is  $w$ -well-dominated. ■

For every graph  $G$ , we denote the vector space of weight functions  $w$  such that  $G$  is  $w$ -well-dominated by  $WWD(G)$ .

Let  $G$  be a graph, and let  $w : V(G) \rightarrow \mathbb{R}$ . Denote  $mDS_w(G)$ ,  $MDS_w(G)$ ,  $mIS_w(G)$ ,  $MIS_w(G)$  the minimum weight of a dominating set, the maximum

weight of a minimal dominating set, the minimum weight of a maximal independent set, and the maximum weight of an independent set, respectively.

The fact that every maximal independent set is also a minimal dominating set implies that

$$mDS_w(G) \leq mIS_w(G) \leq MIS_w(G) \leq MDS_w(G)$$

for every graph  $G$  and every weight function  $w$  defined on its vertices.

If  $mIS_w(G) = MIS_w(G)$  then  $G$  is  $w$ -well-covered, and if  $mDS_w(G) = MDS_w(G)$  then  $G$  is  $w$ -well-dominated. Theorem 2 is an instance of the following.

**Corollary 8** *For every graph  $G$  and for every weight function  $w : V(G) \rightarrow \mathbb{R}$ , if  $G$  is  $w$ -well-dominated then  $G$  is  $w$ -well-covered, i.e.,  $WWD(G)$  is a subspace of  $WCW(G)$ .*

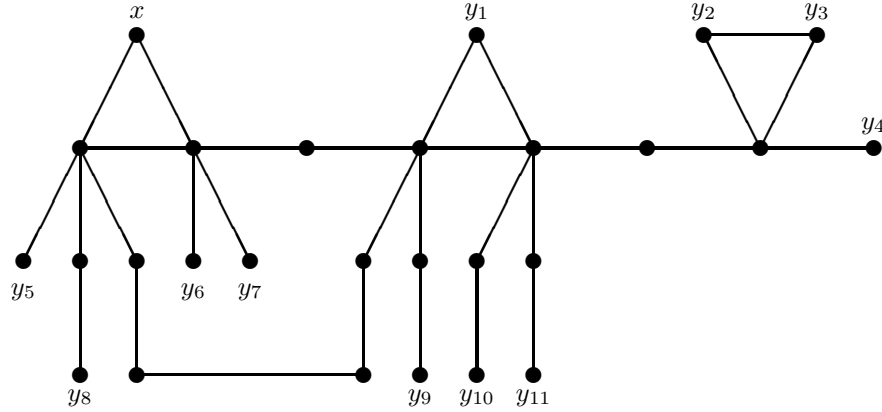


Figure 2: An example of the definition of  $L^*(G)$ . In this graph,  $G$ , it holds that  $L^*(G) = \{y_1, \dots, y_{11}\}$  and  $L(G) \setminus L^*(G) = \{x\}$ . Let  $w : V(G) \rightarrow \mathbb{R}$ . By Theorem 1,  $G$  is  $w$ -well-covered if and only if  $w(y_2) = w(y_3)$  and  $w(v) = w(M(v))$  for every  $v \in V(G) \setminus L(G)$ . By Theorem 9,  $G$  is  $w$ -well-dominated if and only if  $G$  is  $w$ -well-covered and  $w(x) = 0$ .

Let  $L^*(G)$  be the set of all vertices  $v \in V(G)$  such that either

- $d(v) = 1$ ;
- or
- the following conditions hold:
  - $d(v) = 2$ ;
  - $v$  is on a triangle,  $(v, v_1, v_2)$ ;

- Every maximal independent set of  $V(G) \setminus N_2[v]$  dominates at least one of  $N(v_1) \cap N_2(v)$  and  $N(v_2) \cap N_2(v)$ .

Note that  $L^*(G) \subseteq L(G)$  (see Figure 2). Moreover,  $v \in L(G) \setminus L^*(G)$  if and only if the following conditions hold:

- $d(v) = 2$
- $v$  is on a triangle,  $(v, v_1, v_2)$ .
- There exists a maximal independent set of  $V(G) \setminus N_2[v]$  which dominates neither  $N(v_1) \cap N_2(v)$  nor  $N(v_2) \cap N_2(v)$ .

**Theorem 9** *Let  $G \in \mathcal{G}(\widehat{C_4}, \widehat{C_5}, \widehat{C_6})$  be a connected graph, and let  $w : V(G) \rightarrow \mathbb{R}$ . Then  $G$  is  $w$ -well-dominated if and only if one of the following holds:*

1.  $G$  is isomorphic to either  $C_7$  or  $T_{10}$  (see Figure 1), and there exists a constant  $k \in \mathbb{R}$  such that  $w \equiv k$ .
2. The following conditions hold:
  - (a)  $G$  is isomorphic to neither  $C_7$  nor  $T_{10}$ .
  - (b) For every two vertices,  $l_1$  and  $l_2$ , in the same component of  $L(G)$  it holds that  $w(l_1) = w(l_2)$ .
  - (c)  $w(v) = 0$  for every vertex  $v \in L(G) \setminus L^*(G)$ .
  - (d) For every  $v \in V(G) \setminus L(G)$  it holds that  $w(v) = w(M(v))$  for some maximal independent set  $M(v)$  of  $D(v)$ .

**Proof.** The following cases are considered.

CASE 1:  $G$  IS ISOMORPHIC TO EITHER  $C_7$  OR  $T_{10}$ . If there does not exist a constant  $k \in \mathbb{R}$  such that  $w \equiv k$ , then by Theorem 1,  $G$  is not  $w$ -well-covered. By Corollary 8,  $G$  is not  $w$ -well-dominated. Suppose that  $w \equiv k$  for some  $k \in \mathbb{R}$ . If  $G$  is isomorphic to  $C_7$ , then the cardinality of every minimal dominating set is 3. Hence,  $mDS_w(C_7) = MDS_w(C_7) = 3k$ , and  $G$  is  $w$ -well-dominated. If  $G$  is isomorphic to  $T_{10}$ , then the cardinality of every minimal dominating set is 4. Hence,  $mDS_w(T_{10}) = MDS_w(T_{10}) = 4k$ , and  $G$  is  $w$ -well-dominated.

CASE 2:  $L(G) = V(G)$ . In this case  $G$  is a complete graph with at most 3 vertices. In that case the cardinality of every minimal dominating set is 1. Therefore,  $G$  is  $w$ -well-dominated if and only if there exists a constant  $k \in \mathbb{R}$  such that  $w \equiv k$ . In this case  $mDS_w(G) = MDS_w(G) = k$ .

CASE 3:  $L(G) \neq V(G)$  AND  $G$  IS ISOMORPHIC TO NEITHER  $C_7$  NOR  $T_{10}$ . Let  $N(L^*(G)) = \{v_1, \dots, v_k\}$ . Then

$$L^*(G) \subseteq \bigcup_{1 \leq i \leq k} D(v_i) \subseteq L(G).$$

For each  $1 \leq i \leq k$  let  $V_i = \{v_i\} \cup D(v_i)$ .

Assume that Condition 2 holds. Let  $S$  be a minimal dominating set of  $G$ . Then  $S \cap V_i$  is either  $v_i$  or  $M(v_i)$ . Hence,  $w(S \cap V_i) = w(v_i)$  for every  $1 \leq i \leq k$ . Let  $1 \leq i < j \leq k$ . Then  $V_i \cap V_j \subseteq L(G) \setminus L^*(G)$ . Hence,  $w(x) = 0$  for each  $x \in V_i \cap V_j$ . The fact that  $w(v) = 0$  for every  $v \in V \setminus N[L(G)]$  implies that

$$\begin{aligned} w(S) &= w(S \setminus (\bigcup_{1 \leq i \leq k} V_i)) + \sum_{1 \leq i \leq k} w(S \cap V_i) - \sum_{1 \leq i < j \leq k} w(S \cap V_i \cap V_j) = \\ &= 0 + \sum_{1 \leq i \leq k} w(v_i) - 0 = \sum_{1 \leq i \leq k} w(v_i). \end{aligned}$$

Hence,

$$mDS_w(G) = MDS_w(G) = \sum_{1 \leq i \leq k} w(v_i),$$

and  $G$  is  $w$ -well-dominated.

Assume that  $G$  is  $w$ -well-dominated. Then, by Corollary 8,  $G$  is  $w$ -well-covered. By Theorem 1, Conditions 2a, 2b and 2d hold. It remains to prove that Condition 2c holds as well. Let  $v \in L(G) \setminus L^*(G)$ . We prove that  $w(v) = 0$ . Let  $N(v) = \{v_1, v_2\}$ , and let  $S$  be a maximal independent set of  $G \setminus N_2[v]$  which dominates neither  $N(v_1) \cap N_2(v)$  nor  $N(v_2) \cap N_2(v)$ . For each  $1 \leq i \leq 2$  let  $S_i$  be a maximal independent set of  $(N(v_i) \cap N_2(v)) \setminus N(S)$ . Define  $T_i = S \cup S_{2-i} \cup \{v_i\}$  for  $i = 1, 2$ . Define  $T_3 = S \cup S_1 \cup S_2 \cup \{v\}$  and  $T_4 = S \cup \{v_1, v_2\}$ . Clearly,  $T_1, T_2, T_3$  and  $T_4$  are minimal dominating sets of  $G$ .

For each  $i = 1, 2$  the fact that  $w(T_i) = w(T_3)$  implies  $w(S_i \cup \{v\}) = w(v_i)$ . Therefore  $w(S_i) + w(v) = w(v_i)$ . The fact that  $w(T_3) = w(T_4)$  implies  $w(S_1 \cup S_2 \cup \{v\}) = w(\{v_1, v_2\})$ . Therefore,  $w(S_1) + w(S_2) + w(v) = w(v_1) + w(v_2)$ . Hence,  $w(S_1) + w(S_2) + w(v) = w(S_1) + w(S_2) + 2w(v)$ . Thus  $w(v) = 0$ . ■

**Corollary 10**  $\dim(WWD(G)) = |L^*(G)|$  for every graph  $G \in \mathcal{G}(\widehat{C}_4, \widehat{C}_5, \widehat{C}_6)$ .

**Corollary 11** Suppose  $G \in \mathcal{G}(\widehat{C}_4, \widehat{C}_5, \widehat{C}_6)$ . If  $L^*(G) = L(G)$ , then  $WWD(G) = WCW(G)$ . Otherwise,  $WWD(G) \subsetneq WCW(G)$ .

Combining Corollaries 10, 11 with Algorithm 20 from [10] we obtain the following.

**Corollary 12** If  $G \in \mathcal{G}(\widehat{C}_4, \widehat{C}_5, \widehat{C}_6)$ , then

$$|L^*(G)| = \dim(WWD(G)) \leq \dim(WCW(G)) = \alpha(G[L(G)]).$$

Theorem 9 does not hold if  $G \notin \mathcal{G}(\widehat{C}_6)$ . Let  $G$  be the graph with two edge disjoint 6-cycles,  $(v_1, \dots, v_6)$  and  $(v_6, \dots, v_{11})$ . Clearly,  $G \in \mathcal{G}(\widehat{C}_4, \widehat{C}_5)$  and  $L(G) = L^*(G) = \Phi$ . However, the vector space  $WWD(G)$  is the set of all functions  $w : V(G) \rightarrow \mathbb{R}$  which satisfy

1.  $w(v_1) = w(v_2) = -w(v_4) = -w(v_5)$
2.  $w(v_7) = w(v_8) = -w(v_{10}) = -w(v_{11})$
3.  $w(v_3) = w(v_6) = w(v_9) = 0$



## 4 Future Work

The main findings of the paper stimulate us to discover more cases, where the  $WD$  and/or  $WWD$  problems can be solved polynomially.

We have proved that if  $G \in \mathcal{G}(\widehat{C}_4, \widehat{C}_5)$ , then  $G$  is well-dominated if and only if it is well-covered. It motivates the following.

**Problem 13** *Characterize all graphs, which are both well-covered and well-dominated.*

We have also shown that if  $G \in \mathcal{G}(\widehat{C}_4, \widehat{C}_5, \widehat{C}_6)$  and  $L^*(G) = L(G)$ , then  $WCW(G) = WWD(G)$ . Thus one may be interested in approaching the following.

**Problem 14** *Characterize all graphs, where the equality  $WCW(G) = WWD(G)$  holds.*

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